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# On Klauder's pseudo-free oscillator

Bernard S Kay†
Blackett Laboratory, Imperial College, London SW7 2BZ, UK

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**Abstract.** We study the quantum oscillator on the half-line defined by the Hamiltonian $H=-\frac{1}{2}\partial^2/\partial x^2+\frac{1}{2}x^2$ together with vanishing boundary conditions at 0. We derive and discuss a new operator equation of motion of the form $\ddot{x}+x=A$ for this system. This equation is used to verify a conjecture (due to Klauder) that $x$ satisfy $x(\ddot{x}+x)=0$. It is also used to obtain systematic results on the set of $n$-point functions $\langle\Omega|x(t_1)\,x(t_2)\ldots x(t_n)|\Omega\rangle$. It is suggested that our methods and results will act as a source of conjectures about the analogous pseudo-free field theories postulated by Klauder in his program for handling non-renormalisable field theories.

## 1. Introduction

Various forms of (harmonic and anharmonic) oscillator have long served as simple analogue systems for relativistic quantum field theories.

In Klauder's general approach (Klauder 1979) to non-renormalisable quantum field theories, the *pseudo-free oscillator* serves as an important illustrative example.

The pseudo-free oscillator Hamiltonian is just the usual harmonic oscillator Hamiltonian

$$H=-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{x^2}{2} \tag{1}$$

restricted to the half-line (i.e. to $L^2(0,\infty)$)—with vanishing boundary conditions imposed at 0.

This system arises (Klauder 1973) when one perturbs the free harmonic oscillator by highly singular terms like $\lambda|x|^{-\alpha}$ $(\alpha>2)$. In these cases, the system returns to the pseudo-free rather than to the free system as one switches off the coupling $\lambda$.

Klauder's program postulates an analogous behaviour for non-renormalisable interactions. As one switches off the coupling, they are expected to return not to the free theory but rather to some—as yet unknown—pseudo-free theory.

Klauder (1979) makes a strong case for the existence of pseudo-free field theories, and several different (and mutually consistent) lines of attack are suggested for finding them.

These pseudo-free theories are, of course, extreme limiting cases of highly interacting theories, and as such are far from being free themselves.

† Present address: Institut für Theoretische Physik, Universität Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland.

  

Nevertheless, precisely because they constitute some sort of ideal limiting case one might hope for certain special properties and simplifications which would make it possible to solve them exactly at least for some of their properties.

Once such an exact solution was at hand, the prospect would open up of being able to treat non-renormalisable interactions in perturbation theory using the pseudo-free theory as a starting point. In the absence of any ready solution, it seems worthwhile to exploit to the full the analogy between the (unknown and difficult) pseudo-free field theories and the (known and simple) pseudo-free oscillator in order to get at least some idea as to how an exact solution might look.

In this paper we obtain information about the set of $n$-point functions for the pseudo-free oscillator—this being the sort of information one wants to have in a field theory.

There are of course many differences between the pseudo-free oscillator and the full infinite-degrees-of-freedom problem. The pseudo-free oscillator is designed to cope with trouble caused by interactions singular at small $x$ values, whereas the pseudo-free field theory has to cope with trouble at large field configurations. Again, in the field theory case, one expects (Klauder 1979) infinite multiplicative field renormalisations to play a crucial role. These have no analogue with only one degree of freedom.

Nonetheless, there are two (related) points of close similarity (Klauder 1979):
(i) One expects the (say scalar) pseudo-free field theory to obey the equation

$$\phi(\Box+m^2)\phi=0. \tag{2}$$

One expects the pseudo-free oscillator to obey the analogous equation

$$x(\ddot{x}+x)=0. \tag{3}$$

(This latter equation will be proved in § 2 below).
(ii) One expects the (*truncated*) Green's functions of the (scalar) pseudo-free field theory to obey—in some sense—the coupled equations

$$\mathrm{i}\left[\sum_{r=1}^{m}\delta(x-x_r)\right]g(x_1,\ldots,x_m)+(\Box_x+m^2)g(x,x^*,x_1,\ldots,x_m)=0 \tag{4}$$

(here $x^*$ is spacelike separated with respect to $x$ and set equal to $x$ after the differentiation). One *knows* (it follows easily from (3)) that the (now *not* truncated) Green's functions of the pseudo-free oscillator satisfy the exactly analogous equations

$$\mathrm{i}\left[\sum_{r=1}^{m}\delta(t-t_r)\right]\langle\Omega|T(x(t_1)x(t_2)\ldots x(t_m))|\Omega\rangle$$
$$+\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+1\right)\langle\Omega|T(x(t)x(t^*)x(t_1)\ldots x(t_m))|\Omega\rangle=0. \tag{5}$$

(Here the time-ordering instruction must be understood as treating $x(t)x(t^*)$ as a single object ordered according to $t$, and $t^*$ is set equal to $t$ after the differentiation).

The plan of this paper is as follows. In § 2 we collect together some preliminary facts about the pseudo-free oscillator. In § 3 we establish a new equation of motion for $x$. This equation is shown to imply (3) (it contains more information than (3)). In § 4 our new equation of motion is exploited to obtain information about the $n$-point functions. In § 5 we speculate on the possible relevance of our results to the field-theory case. Finally, we sketch—in the Appendix—an alternative more 'physical' derivation of our new equation of motion.

## 2. Preliminaries

Firstly, we recall that some caution is necessary when working with quantum mechanics on the half-line. The usual 'heuristic' approach can sometimes mislead. For example, the momentum operator

$$p = -\mathrm{i}\partial/\partial x \tag{6}$$

(defined say on $C_0^\infty(0, \infty)$) has no self-adjoint extensions (on $L^2(0, \infty)$) and in consequence does not generate a one-parameter family of unitary operators. Another example of 'counter-heuristic' behaviour is given in § 3.

The Hamiltonian operator (1) however is perfectly well defined ($H$ on $C_0^\infty(0, \infty)$ has several self-adjoint extensions but there is precisely one which corresponds to vanishing boundary conditions at 0 (Reed and Simon 1975)).

In fact, a complete set of normalised eigenfunctions is readily found simply by picking out the odd eigenfunctions for the usual harmonic oscillator (i.e. on the full line) (see e.g. Messiah 1968) and normalising to the half-line.

Thus we have

$$H\Psi_a = (2a + \tfrac{3}{2})\Psi_a \qquad a = 0, 1, 2, \ldots \tag{7}$$

where

$$\Psi_a(x) = 2^{1/2}[\pi^{1/2}2^{2a+1}(2a+1)!]^{-1/2} H_{2a+1}(x) \exp(-x^2/2). \tag{8}$$

($H_n(z)$ are the usual Hermite polynomials.)

We shall denote the ground state ($\Psi_0(x) = 2\pi^{-1/4}x \exp(-x^2/2)$) by $\Omega$—it corresponds to the first excited state of the usual harmonic oscillator.

In the sequel, we shall also require the Feynman propagator

$$K(x_1, t_1; x_2, t_2) = \langle x_1|\exp[-\mathrm{i}H(t_1 - t_2)]|x_2\rangle \tag{9}$$

which, again, is readily found (Klauder 1973) to be given by

$$K(x_1, t_1; x_2, t_2) = G(x_1, t_1; x_2, t_2) - G(x_1, t_1; -x_2, t_2) \tag{10}$$

where $G(x_1, t_1; x_2, t_2)$ is the Feynman propagator for the usual harmonic oscillator (Feynman and Hibbs 1965)

$$G(x_1, t_1; x_2, t_2) = \left(\frac{1}{2\pi\,\mathrm{i}\sin t}\right)^{1/2} \exp\left\{\frac{\mathrm{i}}{2\sin t}[(x_1^2 + x_2^2)\cos t - 2x_1x_2]\right\} \tag{11}$$

where $t = t_1 - t_2$.

For later convenience we note the following simple consequences of (10):

$$\frac{\partial}{\partial x'}K(x, t; x', 0)\bigg|_{x'=0} = \frac{\partial}{\partial x'}K(x', t; x, 0)\bigg|_{x'=0} = \left(\frac{2}{\pi}\right)^{1/2} \frac{x}{(\mathrm{i}\sin t)^{3/2}} \exp\left\{-\frac{x^2}{2}\left(\frac{\cos t}{\mathrm{i}\sin t}\right)\right\}$$

$$\frac{\partial^2}{\partial x\partial x'}K(x, t; x', 0)\bigg|_{x=x'=0} = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(\mathrm{i}\sin t)^{3/2}}. \tag{12}$$

Finally, since our main interest—pursuing the analogy with field theory—is in the $n$-point functions $\langle\Omega|x(t_1)x(t_2)\ldots x(t_n)|\Omega\rangle$ we record what can be obtained by elementary methods. For $n = 1$, we easily have

$$\langle\Omega|x(t)|\Omega\rangle = 2\pi^{-1/2}. \tag{13}$$

For $n=2$, we have

$$\langle\Omega|x(t_1)x(t_2)|\Omega\rangle = \sum_{a=0}^{\infty} |\langle\Omega|x(t)|\Psi_a\rangle|^2 \exp[-2\mathrm{i}a(t_1-t_2)] \tag{14}$$

which, by elementary properties of Hermite polynomials, sums to

$$(4/\pi)F(-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{3}{2}; \exp[-2\mathrm{i}(t_1-t_2)])$$

where $F$ is the hypergeometric function. In terms of elementary functions, this is

$$\frac{4}{\pi}\left\{\left(\frac{\exp[-\mathrm{i}(t_1-t_2]}{2}+\frac{\exp[\mathrm{i}(t_1-t_2)]}{4}\right)\sin^{-1}(\exp[-\mathrm{i}(t_1-t_2)])+\frac{3}{4}(1-\exp[-2\mathrm{i}(t_1-t_2)])^{1/2}\right\}. \tag{15}$$

Beyond $n=2$, it no longer appears practical to continue in this way and we are encouraged to look for another method.

## 3. Equation of motion

If we were to suspend for a moment our caution about working on the half-line, we might expect $x(t)$ to satisfy the equation of motion

$$\ddot{x}(t)+x(t)=0. \tag{16}$$

Equation (16) would appear to follow on writing

$$H=p^2/2+x^2/2 \tag{17}$$

and using the usual CCR's $[x, p]=\mathrm{i}$ and Hamiltonian equations $\dot{x}=\mathrm{i}[H, x]$, $\dot{p}=\mathrm{i}[H, p]$.

However equation (16) is false. For instance, it would imply $\langle\Omega|x(t)|\Omega\rangle=0$ in contradiction with equation (13).

In order to obtain the correct equation of motion, we must be a little more careful about domain questions.

Choose the domain D consisting of smooth square-integrable functions vanishing at 0. (D has the virtue of containing all the energy eigenstates.)

Then, working in the Schrödinger picture, with

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{x^2}{2}\right)\Psi=\mathrm{i}\frac{\partial\Psi}{\partial t} \tag{18}$$

one obtains

$$(\partial/\partial t)\langle\Psi|x|\Phi\rangle=\langle\Psi|p|\Phi\rangle \tag{19}$$

where $p=-\mathrm{i}\partial/\partial x$ and

$$(\partial/\partial t)\,\langle\Psi|p|\Phi\rangle=-\langle\Psi|x|\Phi\rangle+\tfrac{1}{2}\,\Psi^{*\prime}(0)\Phi'(0). \tag{20}$$

It is the last term here—which arises as a boundary term at 0—which was missing in our earlier 'incautious' discussion. We can take it into account by defining the 'operator' (actually quadratic form (Reed and Simon 1972)):

$$\langle\Psi|A|\Phi\rangle=\tfrac{1}{2}\Psi^{*\prime}(0)\Phi'(0) \qquad \Phi, \Psi\in \mathrm{D} \tag{21}$$

whereupon we have the corrected equation of motion (transforming back to the

Heisenberg picture)

$$\ddot{x}(t)+x(t)=A(t) \tag{22}$$

where $A(t)=\exp(\mathrm{i}Ht)A\exp(-\mathrm{i}Ht)$. ((Equation 22) is to be interpreted as

$$(\mathrm{d}^2/\mathrm{d}t^2+1)\langle\Psi|x(t)|\Phi\rangle=\langle\Psi|A(t)|\Phi\rangle \qquad \forall\Psi,\Phi\in \mathrm{D}.)$$

We sketch an alternative, more 'physical' derivation of equation (22) in the Appendix. Klauder's conjectured equation

$$x(\ddot{x}+x)=0 \tag{3}$$

is then immediately seen to hold in the sense that

$$(\mathrm{d}^2/\mathrm{d}t^2+1)\langle\Psi|x(t^*)x(t)|\Phi\rangle=0 \qquad \forall\Phi,\Psi\in \mathrm{D} \tag{23}$$

where $t^*$ is set equal to $t$ after the differentiation.

Finally, it is interesting to compare the above with the *classical* situation. Classically, a harmonic oscillator which is constrained to 'bounce' at the origin will have solutions (Klauder 1979)

$$x=A|\cos(t+B)|. \tag{24}$$

Clearly, these obey equation (3). If one seeks an equation like (22) one readily obtains (in some formal sense)

$$\ddot{x}+x=2p^2\delta(x) \tag{25}$$

where $p=\dot{x}$. (It does not seem possible to establish any exact correspondence between equations (22) and (25) although it is tempting to try to write $A$ as something like $\frac{1}{2}p\delta(x)p$.) Equations like (25) have previously been considered by Castell (1978).

## 4. Information about $n$-point functions

One can exploit equation (22) to obtain information about the $n$-point functions. In a basis that diagonalises $x(t)$, we have, from (21),

$$\langle x\,t|A(t)|x'\,t\rangle=\tfrac{1}{2}\delta'(x)\delta'(x') \tag{26}$$

(where $\delta'(x)$ is interpreted (unconventionally) as yielding $\int_0^\infty f(x)\delta'(x)\,\mathrm{d}x=-f'(0)$ for $f$ with $f(0)=0$). This combines neatly with the results (12) on the Feynman propagator given in § 1 to allow the following calculation:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t_1^2}+1\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}t_2^2}+1\right)\cdots\left(\frac{\mathrm{d}^2}{\mathrm{d}t_n^2}+1\right)\langle\Omega|x(t_1)x(t_2)\ldots x(t_n)|\Omega\rangle$$

$$=\langle\Omega|A(t_1)A(t_2)\ldots A(t_n)|\Omega\rangle$$

$$=\int \mathrm{d}x_1\int \mathrm{d}x_1'\int \mathrm{d}x_2\int \mathrm{d}x_2'\cdots\int \mathrm{d}x_n\int \mathrm{d}x_n'\,\langle\Omega|x_1t_1\rangle\langle x_1t_1|A(t_1)|x_1't_1\rangle$$

$$\times\langle x_1't_1|x_2t_2\rangle\langle x_2t_2|A(t_2)|x_2't_2\rangle\langle x_2't_2|x_3t_3\rangle\ldots$$

$$\ldots\langle x_{n-1}'t_{n-1}|x_nt_n\rangle\langle x_nt_n|A(t_n)|x_n't_n\rangle\langle x_n't_n|\Omega\rangle$$

$$= \int dx_1 \int dx'_1 \int dx_2 \int dx'_2 \ldots \int dx_n \int dx'_n\, (\exp(3it/2)2\pi^{-1/4}x_1$$

$$\times \exp(-x_1^2/2))(\tfrac{1}{2}\delta'(x_1)\delta'(x'_1))K(x'_1, t_1; x_2, t_2)(\tfrac{1}{2}\delta'(x_2)\delta'(x'_2))$$

$$\times K(x'_2, t_2; x_3, t_3) \ldots K(x_{n-1}, t_{n-1}; x_n, t_n)$$

$$\times (\tfrac{1}{2}\delta'(x_n)\delta'(x'_n))\, (\exp(-3it_n/2)2\pi^{-1/4}x'_n \exp(-x'^2_n/2))$$

$$= (\exp(3it_1/2)2\pi^{-1/4}) \left(\frac{1}{2}\right) \left[\left(\frac{2}{\pi}\right)^{1/2} \frac{1}{[\mathrm{i}\sin(t_1-t_2)]^{3/2}}\right] \left(\frac{1}{2}\right)\left[\left(\frac{2}{\pi}\right)^{1/2}\right.$$

$$\left.\times \frac{1}{[\mathrm{i}\sin(t_2-t_3)]^{3/2}}\right] \ldots \left[\left(\frac{2}{\pi}\right)^{1/2} \frac{1}{[\mathrm{i}\sin(t_{n-1}-t_n)]^{3/2}}\right]$$

$$\times \left(\frac{1}{2}\right) (\exp(-3it_n/2)2\pi^{-1/4})$$

giving finally

$$\langle\Omega|A(t_1)A(t_2)\ldots A(t_n)|\Omega\rangle$$

$$= 2\pi^{-1/2}\left(\frac{1}{(2\pi)^{1/2}}\right)^{n-1} (\mu(t_1-t_2)\mu(t_2-t_3)\ldots\mu(t_{n-1}-t_n))^{-3/2} \tag{27}$$

where

$$\mu(t) = \exp(-\mathrm{i}t)\sin t = (1-\exp(-2\mathrm{i}t))/2.$$

With a little more work, one may eliminate alternate factors of $(\mathrm{d}^2/\mathrm{d}t_i^2+1)$ in the above:

$$\left(\frac{d^2}{dt_2^2}+1\right)\left(\frac{d^2}{dt_4^2}+1\right)\ldots\left(\frac{d^2}{dt_{[n]}^2}+1\right)\langle\Omega|x(t_1)x(t_2)\ldots x(t_n)|\Omega\rangle$$

($[n]$ is $n$ when $n$ even, $n-1$ when $n$ odd)

$$= \langle\Omega|x(t_1)A(t_2)x(t_3)\ldots A(t_n)|\Omega\rangle \qquad (n \text{ even})$$

$$\langle\Omega|x(t_1)A(t_2)x(t_3)\ldots x(t_n)|\Omega\rangle \qquad (n \text{ odd}).$$

Taking the $n$ even case:

$$= \int dx_1 \iint dx_2\, dx'_2 \int dx_3 \iint dx_4\, dx'_4 \ldots \iint dx_n\, dx'_n \langle\Omega|x_1t_1\rangle$$

$$\times \langle x_1t_1|x(t_1)|x_2t_2\rangle\langle x_2t_2|A(t_2)|x'_2t_2\rangle\langle x'_2t_2|x(t_3)|x_3t_3\rangle$$

$$\times \langle x_3t_3|x_4t_4\rangle\langle x_4t_4|A(t_4)|x'_4t_4\rangle \ldots$$

$$\ldots \langle x_{n-1}t_{n-1}|x_nt_n\rangle\langle x_nt_n|A(t_n)|x'_nt_n\rangle\langle x'_nt_n|\Omega\rangle$$

$$= \int dx_1 \iint dx_2\, dx'_2 \int dx_3 \iint dx_4\, dx'_4 \ldots$$

$$\ldots \iint dx_n\, dx'_n\, (\exp(3it_1/2)2\pi^{-1/4}x_1 \exp(-x_1^2/2))x_1$$

$$\times K(x_1, t_1; x_2, t_2)\tfrac{1}{2}\delta'(x_2)\delta'(x_2')K(x_2', t_2; x_3, t_3)x_3K(x_3, t_3; x_4, t_4)$$
$$\times\tfrac{1}{2}\delta'(x_4)\ldots K(x_{n-1}, t_{n-1}; x_n, t_n)$$
$$\times\left(\frac{1}{2}\right)\delta'(x_n)\delta'(x_n')\,(\exp(-3\mathrm{i}t_n/2)2\pi^{-1/4}x_n'\exp(-x_n'^2/2))$$
$$=\int \mathrm{d}x_1\int \mathrm{d}x_3\int \mathrm{d}x_5\ldots\int \mathrm{d}x_{n-1}(\exp(3\mathrm{i}t_1/2)2\pi^{-1/4}x_1^2\exp(-x_1^2/2))$$
$$\times\left\{\left(\frac{2}{\pi}\right)^{1/2}\frac{x_1}{[\mathrm{i}\sin(t_1-t_2)]^{3/2}}\exp\left[-\frac{x_1^2}{2}\left(\frac{\cos(t_1-t_2)}{\mathrm{i}\sin(t_1-t_2)}\right)\right]\right\}\left(\frac{1}{2}\right)$$
$$\times\left\{\left(\frac{2}{\pi}\right)^{1/2}\frac{x_3}{[\mathrm{i}\sin(t_2-t_3)]^{3/2}}\exp\left[-\frac{x_3^2}{2}\left(\frac{\cos(t_2-t_3)}{\mathrm{i}\sin(t_2-t_3)}\right)\right]\right\}$$
$$\times x_3\left\{\left(\frac{2}{\pi}\right)^{1/2}\frac{x_3}{[\mathrm{i}\sin(t_3-t_4)]^{3/2}}\exp\left[-\frac{x_3^2}{2}\left(\frac{\cos(t_3-t_4)}{\mathrm{i}\sin(t_3-t_4)}\right)\right]\right\}\left(\frac{1}{2}\right)$$
$$\times\left\{\left(\frac{2}{\pi}\right)^{1/2}\frac{x_{n-1}}{[\mathrm{i}\sin(t_{n-1}-t_n)]^{3/2}}\exp\left[-\frac{x_{n-1}^2}{2}\left(\frac{\cos(t_{n-1}-t_n)}{\mathrm{i}\sin(t_{n-1}-t_n)}\right)\right]\right\}$$
$$\times\left(\frac{1}{2}\right)(\exp(-3\mathrm{i}t_n/2)2\pi^{-1/4})$$
$$=(\exp(3\mathrm{i}t_1/2)2\pi^{-1/4})\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{1}{2}\right)\int \mathrm{d}x_1\frac{x_1^3}{[\mathrm{i}\sin(t_1-t_2)]^{3/2}}$$
$$\times\exp\left[-\frac{x_1^2}{2}\left(1+\frac{\cos(t_1-t_2)}{\mathrm{i}\sin(t_1-t_2)}\right)\right]$$
$$\times\left(\frac{2}{\pi}\right)\left(\frac{1}{2}\right)\int \mathrm{d}x_3\frac{x_3^3}{[\mathrm{i}\sin(t_2-t_3)]^{3/2}[\mathrm{i}\sin(t_3-t_4)]^{3/2}}$$
$$\times\exp\left[-\frac{x_3^2}{2}\left(\frac{\cos(t_2-t_3)}{\mathrm{i}\sin(t_2-t_3)}+\frac{\cos(t_3-t_4)}{\mathrm{i}\sin(t_3-t_4)}\right)\right]$$
$$\text{etc}\qquad\int \mathrm{d}x_5\qquad\text{etc}\ldots(\exp(-3\mathrm{i}t_n/2)2\pi^{-1/4}).$$

On using $\int_0^\infty \mathrm{d}x\, x^3\exp(-ax^2)=(1/2a^2)$ this yields

$$(\exp(3\mathrm{i}t_1/2)2\pi^{-1/4})\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{1}{2}\right)\frac{1}{[\mathrm{i}\sin(t_1-t_2)]^{3/2}}(2\exp[-2\mathrm{i}(t_1-t_2)])[\mathrm{i}\sin(t_1-t_2)]^2$$
$$\times\left(\frac{2}{\pi}\right)\left(\frac{1}{2}\right)\left(\frac{1}{[\mathrm{i}\sin(t_2-t_3][(\mathrm{i}\sin(t_3-t_4)]}\right)^{3/2}$$
$$\times(2)\left(\frac{[\mathrm{i}\sin(t_2-t_3)][\mathrm{i}\sin(t_3-t_4)]}{[\mathrm{i}\sin(t_2-t_4)]}\right)^2\ldots\qquad(\exp(-3\mathrm{i}t_n/2)\,2\pi^{-1/4})$$
$$=4\pi^{-1/2}\left(\frac{2}{\pi}\right)^{(n-1)/2}\frac{[\mu(t_1-t_2)\mu(t_2-t_3)\ldots\mu(t_{n-1}-t_n)]^{1/2}}{[\mu(t_2-t_4)\mu(t_4-t_6)\ldots\mu(t_{n-2}-t_n)]^2}\tag{28a}$$

where $\mu$ is defined after equation (27).

Similarly, for odd $n$ ($n>2$) we obtain

$$8\pi^{-1/2}\left(\frac{2}{\pi}\right)^{(n-1)/2}\frac{[\mu(t_1-t_2)\mu(t_2-t_3)\ldots\mu(t_{n-1}-t_n)]^{1/2}}{[\mu(t_2-t_4)\mu(t_4-t_6)\ldots\mu(t_{n-3}-t_{n-1})]^2}. \tag{28b}$$

As for the (undifferentiated) $n$-point functions themselves, the two- and three-point functions may be deduced from (28) by integrating up with the retarded Green function for $d^2/dt^2+1$, thus for instance

$$\langle\Omega|x(t)x(0)|\Omega\rangle=\frac{4\sqrt{2}}{\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{t}\frac{\sin(t-t')}{2}\exp[-\epsilon(t-t')]\exp(-\mathrm{i}t'/2)[\mathrm{i}\sin t']^{1/2}\,dt' \tag{29}$$

which (after a short calculation) checks with equation (15). However, for $n>3$ such Green function methods fail because of the singular denominator in equation (28) and we have not succeeded in finding a nice general formula for the $n$-point function itself. The form of the two-point function of equation (15) is complicated enough to suggest that such a general formula would be of dubious value anyway!

## 5. Discussion

Our hope is that the methods and results described above will act as a source of conjectures about the solution to pseudo-free field theories.

Equations (4) and (5) suggest that there should be a close analogy between the $n$-point functions of the pseudo-free oscillator and the *truncated* $n$-point functions of the field theory (that is, the generating functional for the oscillator should be like the *logarithm* of the generating functional for the field theory—this difference can be traced back (Klauder 1979) to the need for infinite multiplicative renormalisations in the field theory case).

The lesson of § 3 is that the $n$-point functions themselves are not so nice, but the result of $n$-fold action of the free equation (27) *is* nice.

In other words, we conjecture that

$$(\Box_1+m^2)(\Box_2+m^2)\ldots(\Box_n+m^2)\langle 0|\phi(x_1)\phi(x_2)\ldots\phi(x_n)|0\rangle$$

may be calculable in analogy with the results in this paper.

It is intriguing to note that (apart from time ordering, which can presumably be dealt with by 'Euclideanising') this is exactly what one needs (e.g. Bjorken and Drell 1965) (the so-called 'on-mass-shell amputated Green's function') to calculate the $S$ matrix!

Finally, we should not lose sight of the fact that the pseudo-free oscillator—and its use as a starting point for singular perturbations in quantum mechanics—is not without interest in its own right (Klauder 1973, Ezawa *et al* 1975).

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## Appendix

We present an alternative heuristic understanding of equation (22). We regard our (half-line) problem as a limiting case of a quantum mechanical problem on the (full) line. Take the Hamiltonian

$$H = p^2/2 + x^2/2 + h\theta(-x) \tag{A1}$$

where $\theta$ is the step function. (The potential $V(x) = x^2/2 + h\theta(-x)$ is sketched in figure 1.) Our problem is recovered in the limit $h \to \infty$ (infinite step). Since we are working on the full-line we can safely assume the expected equations of motion will hold. Thus we shall have

$$\ddot{x} + x = h\delta(x). \tag{A2}$$

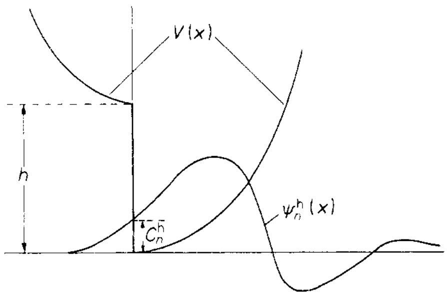


**Figure 1.** The potential and a typical energy eigenstate for the finite-barrier approximation.

We shall 'sandwich' (A2) between two eigenstates $\Psi_n^{\mathrm{h}}$, $\Psi_m^{\mathrm{h}}$. The superscript h reminds us that these states depend on the height of the step. When the step is large but finite, we expect the eigenstates to look broadly like those for the limiting ($h \to \infty$) system, except for an exponential tail on the left (see figure 1). Thus

$$\text{(A2)} \Rightarrow \langle \Psi_n^{\mathrm{h}} | \ddot{x} + x | \Psi_m^{\mathrm{h}} \rangle = C_n^{\mathrm{h}*} C_m^{\mathrm{h}} h \tag{A3}$$

where $C_n^{\mathrm{h}}$ is the height of the exponential tail ($= \Psi_n^{\mathrm{h}}(0)$). Now (see figure 1) although the value of $\Psi_n^{\mathrm{h}}$ at zero is not the same as that of the limiting case, one *does* feel safe (see figure 1) in approximating the *gradient* $\Psi_n^{\mathrm{h}'}(0)$ by that of the limiting case.

If we accept this, then, applying the usual 'matching' conditions of the wavefunction and its gradient at 0, and assuming $h \gg E_n, E_m$, we easily find $C_n^{\mathrm{h}} \approx \Psi_n'(0)/(2h)^{1/2}$. Putting this in (A3) and taking the limit $h \to \infty$, we obtain

$$\langle \Psi_n | \ddot{x} + x | \Psi_m \rangle = \tfrac{1}{2} \Psi_n^{*\prime}(0) \Psi_m'(0). \tag{A4}$$

Applying the usual 'since it is true in a basis, it is true everywhere' argument, we recover

$$\ddot{x} + x = A. \tag{22}$$

(Since this was proved rigorously in § 2, we now feel happy about the assumptions made in the above argument—it can presumably be made rigorous too.)

The following insight has been gained about the origin of equation (22): It is as if our infinite-barrier system 'remembers' the tunnelling there would have been if the barrier were made finite.

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